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Zeta functions on the non-positive real axis

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Abstract. Physical applications of ζ-function regularisation require explicit knowledge of the relevant ζ function somewhere on the non-positive real axis. We present a simple and very general method for evaluating ζ functions $Z(s) = \Sigma_m \lambda_m^{-s}$ at certain discrete points $s = -np \leqslant 0$, $n = 0, 1, 2, \ldots$, on this axis, where p is defined by the condition that λ_m^p be a polynomial in all summation indices. $Z(-np)$ is shown to be a polynomial in the various parameters on which the 'eigenvalues' λ_m depend. The polynomials $Z(-np)$ generalise the Bernoulli polynomials and possess interesting properties. These statements apply to multidimensional (e.g. Epstein and other) ζ functions, as well as to single-sum ζ functions. Physical applications in quantum field theory are briefly indicated.

Introduction

Theoretical physics increasingly employs ζ functions to assign well defined finite values to divergent series which are understood to represent physical quantities. As a rule, this involves evaluation of the ζ function and/or its derivative at some point on the non-positive real axis, well to the left of the ζ function's abcissa of convergence. This analytic continuation procedure is known as 'ζ-function regularisation'. To convey the idea, we briefly describe two physical problems in which ζ-function regularisation is often employed.

Consider first the zero-point energy of a system of quantum oscillators with discrete frequencies $\{\omega_m\}$. This could be the vacuum energy of a field theory constrained by boundaries or by non-trivial topology (so that the frequencies ω_n are discrete). The zero-point energy of the system is (see, e.g., [1])

$$\mathscr{E} \equiv \tfrac{1}{2} \sum_m \omega_m = \tfrac{1}{2}\zeta(-1|\{\omega_m\}) \tag{1.1}$$

$$\zeta(s|\{\omega_m\}) \equiv \sum_m \omega_m^{-s} \qquad \operatorname{Re} s > B > 0. \tag{1.2}$$

The series (1.2) converges only to the right of the abcissa of convergence for this series, say $\operatorname{Re} s = B > 0$. But $\mathscr{E}$ can be identified as the value at $s = -1$ of the analytic function defined by the series (1.2). The regularisation process works as long as $\zeta(s|\{\omega_m\})$ is meromorphic, with no pole at $s = -1$. Then the formally divergent series (1.1) is assigned the finite value $\zeta(-1|\{\omega_m\})$.

A more common use of ζ functions in field theory is the calculation of functional determinants [2-4] or of one-loop effective potentials (see, e.g. [5, 6]). Given an elliptic operator A (typically a Laplacian or some generalisation thereof) the functional determinant of A can be expressed as

$$\ln \det A = -Z'(0) + Z(0) \ln \mu \tag{1.3}$$

$$Z(s) \equiv \sum_m (\lambda_m/\mu)^{-s}. \tag{1.4}$$

Here the λ_m are the positive eigenvalues of A, assumed to be discrete because of some compactification of spacetime. (For flat spacetime and a continuous set of eigenvalues, $Z(s)$ becomes a distribution.) The scale parameter μ has the dimension of A. A true ζ function $Z(s)$ has no pole at $s=0$, and so (1.3) assigns a finite value to $\ln \det A$. This procedure involves not only the ζ function $Z(s)$, but also its derivative

$$Z'(s) = -\sum_m \ln\left(\frac{\lambda_m}{\mu}\right)\left(\frac{\lambda_m}{\mu}\right)^{-s} \tag{1.5}$$

evaluated at $s=0$. $Z'(s)$ is not a ζ function, but something more complicated—a Dirichlet series. In general, Dirichlet series $D(s)$ may be characterised as 'modified ζ functions':

$$D(s) = \sum_m f_m/\lambda_m^s \tag{1.6}$$

which have additional factors f_m inserted under the sum. ($f_m = -\ln \lambda_m$ in (1.5).) As a rule, Dirichlet series have different properties than their 'parent' ζ function $Z(s)$. But they still define meromorphic functions and can be used for regularisation much like ζ functions.

This paper presents a simple method for the evaluation of very general ζ functions and their associated Dirichlet series at an infinite set of discrete points on the non-positive real axis. Notationally we denote these points by

$$s = -np \qquad n = 0, 1, 2, \ldots. \tag{1.7}$$

Here p is the smallest positive number for which λ_m^p is a *polynomial in each of its summation indices.* (Note that multiple sums may be involved.) Then λ_m^{np}, $n = 0, 1, 2, \ldots$, is also a polynomial in the summation indices. This condition is essential for the success of our method. All of the points (1.7) lie to the left of the abcissa of convergence of any ζ function, and normally one would have to perform explicit analytic continuation to reach them. A major advantage of our method is that one does not have to solve the full analytic continuation problem to evaluate the ζ function at these particular points. Indeed, very simple manipulations are sufficient, even for very complicated ζ functions. The main shortcoming of our method is its restrictiveness. Away from the points (1.7) it ceases to apply in the simple version presented here.

An aspect of considerable importance is that the value obtained for $Z(-np) = \Sigma_m \lambda_m^{np}$ is a polynomial in all of the parameters on which the eigenvalues λ_m depend. Several examples of this are known in the classical literature on ζ functions. We shall establish that this is a very general phenomenon—indeed, one of the defining features of ζ functions.

The two preceding paragraphs apply to multiple-sum ζ functions at a very general level, as well as to single-sum ζ functions. Our procedure in this paper will be to introduce our method with the help of examples for which basically everything is known, and then generalise. Thus we begin with single-sum ζ functions in § 2 and then proceed to multiple-sum ones in § 3.

Actual calculations are done as follows. Under the sum in $Z(-np)=\Sigma_m \lambda_m^{np}$, λ_m^{np} is, by definition, a polynomial in all summation indices. For example, in the single-sum case one has

$$\lambda_m^{np}=c_0 m^{\alpha_0}+c_1 m^{\alpha_1}+\ldots+c_N m^{\alpha_N} \qquad \alpha_k \geqslant 0. \tag{1.8}$$

Our observation in this paper is that, because there are a finite number of terms in λ_m^{np}, the sum over m can be evaluated explicitly:

$$\sum_{m=1}^{\infty} \lambda_m^{np}=c_0\zeta(-\alpha_0)+\ldots+c_N\zeta(-\alpha_N)+\{\text{extra terms}\} \tag{1.9}$$

where $\zeta(s)=\Sigma_1^\infty m^{-s}$ is the Riemann ζ function [7, 8]. Essentially, one is commuting a divergent infinite sum Σ_m through a finite series, and this is manageable. Some extra terms may be generated in the process (because Σ_m is divergent). The only real difficulty is to reliably calculate these extra terms. We show precisely how to do this. The multidimensional case is a straightforward extension of what one does in single-sum problems.

Usually the numbers α_k in (1.8) and (1.9) are integers, and thus one obtains a formula expressing the more complicated ζ function $Z(-np)$ in terms of well known special values of the Riemann ζ function [7, 8]:

$$\begin{aligned}\zeta(-n)&=-\frac{B_{n+1}}{n+1} && n=0,1,2,\ldots\\ &=0 && n=2,4,6,\ldots.\end{aligned} \tag{1.10}$$

A great many ζ functions can be dealt with in this way. In particular, we will show that this is true of the Epstein ζ functions [9], which have the important properties:

$$\sum_{m_i=-\infty}^{\infty}[a_1(m_1+g_1)^2+\ldots+a_N(m_N+g_N)^2]^{-s}\exp[\mathrm{i}(m_1h_1+\ldots+m_Nh_N)]$$

$$=0 \qquad \text{at } s=-1,-2,-3,\ldots \tag{1.11a}$$

$$=0 \qquad \text{at } s=0 \text{ if not all the } g_i \text{ are integers} \tag{1.11b}$$

$$=-\exp[-\mathrm{i}(g_1h_1+\ldots+g_Nh_N)] \qquad \text{at } s=0 \text{ if all the } g_i \text{ are integers (in which case the term } m_i=-g_i \text{ is excluded from the sum on the left-hand side).} \tag{1.11c}$$

Epstein had to solve the full analytic continuation problem to obtain (1.11a–c). We will rederive these same results much more simply by expressing the Epstein ζ functions at $s=0,-1,-2,\ldots$, in terms of the special values (1.10) of $\zeta(s)$.

Epstein ζ functions turn up frequently in quantum field theory because they are constructed from eigenvalues quadratic in the summation indices. These are the eigenvalues of quadratic Lagrangian kinetic terms $A=\partial_1^2+\ldots+\partial_N^2$ in momentum space, for theories defined in an N-dimensional box with periodic boundary conditions. We shall obtain, in addition to (1.11a–c), corresponding results for much more general ζ functions than Epstein's. These include, for example, the ζ functions associated with Lagrangian kinetic terms involving arbitrary powers of the derivatives. These more general ζ functions are what one would use to calculate effective potentials for theories with arbitrary higher-derivative kinetic terms. The author is not aware of any results whatever on such ζ functions or effective potentials.

2. Single-sum ζ functions

One can express [5, 6, 10–13] even very complicated single-sum ζ functions in terms of the Riemann ζ function $\zeta(s)$. One uses the binomial expansion to do this. The result is an explicit formula, giving the single-sum ζ function everywhere in the s plane. In this section, we begin with simple examples, showing how a much abbreviated version of this general procedure can be used at those points (1.7) on the non-positive axis which were defined in the introduction. The only subtle aspect of the method arises from the commutation of sums. This will be carefully explained with the help of known ζ functions in § 2.1. (The commutation problem is very important, and its clarification is the reason why the known material in § 2.1 is included.) Thus equipped, it will be quite simple to obtain new results of a very general nature on single-sum ζ functions (in § 2.2) and multidimensional ζ functions (§ 3).

2.1. Hurwitz and related ζ functions

The Hurwitz ζ function is [7, 10]

$$\begin{aligned}\zeta(s, a) &\equiv \sum_{n=0}^{\infty} (n+a)^{-s} \qquad \mathrm{Re}\, s>1 \\ &= a^{-s} + \sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\binom{-s}{k} a^k m^{-s-k} \\ &= a^{-s} + \sum_{k=0}^{\infty}\binom{-s}{k} a^k \zeta(s+k) \qquad \text{all } s, |a|<1. \end{aligned} \tag{2.1}$$

Here, if $|a|<1$, the binomial series Σ_k converges. For Re $s>1$, so does Σ_m. With both sums convergent, Σ_m can be commuted through Σ_k and evaluated as the Riemann ζ functions $\zeta(s+k)$. The resulting exact series is well defined for all s, so the final line in (2.1) explicitly continues $\zeta(s, a)$ to all s.

When $s=-L$ is a non-positive integer, the final line in (2.1) becomes [10] a Bernoulli polynomial [7, 8]

$$\begin{aligned}\zeta(-L, a) &= -\frac{B_{L+1}(a)}{L+1} \\ &= a^L + \sum_{k=0}^{L}\binom{L}{k} a^k \zeta(k-L) + \left\{-\frac{a^{L+1}}{L+1}\right\} \qquad L=0, 1, 2, \ldots .\end{aligned} \tag{2.2}$$

Here the curly bracket term is the $k=L+1$ term in (2.1). This term contributes because the pole in $\zeta(1)$ cancels the zero in the binomial coefficient $\binom{L}{L+1}$. Explicitly, for $s=-L+\varepsilon$, $k=L+1$:

$$\begin{aligned}\binom{L-\varepsilon}{L+1} a^{L+1}\zeta(1+\varepsilon) &= \frac{\Gamma(L+1)a^{L+1}}{(L+1)!\Gamma(-\varepsilon)}[1/\varepsilon+\gamma+\mathrm{O}(\varepsilon)] \\ &= -\frac{a^{L+1}}{L+1}\end{aligned} \tag{2.3}$$

where the limit $\varepsilon \to 0$ is understood.

Suppose that one did not know (2.1) and needed to evaluate $\zeta(-L, a)$. Then one would write

$$\sum_{m=1}^{\infty}(m+a)^L=\sum_{m=1}^{\infty}\sum_{k=0}^{L}\binom{L}{k}a^k m^{L-k}$$

$$=\sum_{k=0}^{L}\binom{L}{k}a^k\zeta(k-L)+\{\text{extra terms}\}. \tag{2.4}$$

Here the curly bracket $\{\}$ represents all extra terms generated by the commutation of the divergent sum Σ_m through the finite sum Σ_k. In general, there may arise corrections to *any* naive manipulation of series when divergent series are involved [10, 12-15]. The author finds it extremely convenient to *use the symbol* $\{\}$ *to represent and to identify these corrections*, as in (2.4) above and in many other formulae to follow in this paper. *A priori* one does not know what this correction is and it may seem a bit unconventional to employ a symbol for something as loosely defined as this. In practice, however, $\{\}$ is precisely calculable for a very broad range of problems [10, 12-15]. As long as one is confident these corrections can be routinely evaluated, the symbol $\{\}$ may be regarded as representing something well defined. This notation only ceases to be meaningful when the corrections it represents cannot be calculated.

There is a systematic way to find extra terms (for arbitrary s) using Cauchy's theorem [13-15]. We do not want to reiterate the Cauchy method here, but rather to present a simpler alternative which can be used in problems like (2.4). In the latter formula, we pick up the extra term simply by extending the finite sum over k to include the term $k=L+1$, which contains $\zeta(1)$. This triggers the mechanism (2.3), and one ends up with the same extra term as before.

In more general problems we have, in place of $(m+a)^L$ in (2.4).

$$\lambda_m^{np}=\sum_{k=0}^{N}c_k m^{\alpha_k} \qquad \alpha_k\geqslant 0 \tag{2.5}$$

leading to

$$\sum_{m=1}^{\infty}\lambda_m^{np}=\sum_{k=0}^{\infty}c_k\zeta(-\alpha_k)+\{\}. \tag{2.6}$$

Here the extra term can be found by extending the finite sum over k to any integral value of k for which $\alpha_k=-1$ (α_k being regarded as some known function of k) to pick up the contributions from terms containing $\zeta(1)$. If there are no such terms, $\{\}=0$. This is the essence of our method.

As an illustration, consider the ζ function:

$$\sum_{m=1}^{\infty}(m^\alpha+a)^{-s}=\sum_{k=0}^{\infty}\binom{-s}{k}a^k\zeta(\alpha(s+k)) \qquad \alpha>0. \tag{2.7}$$

If $\alpha\neq 1/\text{integer}$, then $\alpha(k-L)=1$ is not possible and there can be no extra term. Thus

$$\sum_{m=1}^{\infty}(m^\alpha+a)^L=\sum_{k=0}^{L}\binom{L}{k}a^k\zeta(\alpha(k-L)). \tag{2.8}$$

This polynomial is associated with the ζ function (2.7) in the same way that the

Bernoulli polynomial is associated with $\zeta(s, a)$, and it generalises the Bernoulli polynomials. (The polynomial (2.8) satisfies identities quite similar to those satisfied by the Bernoulli polynomials.)

Three variations of the preceding analysis deserve mention. The first is the insertion of additional powers of n under the sum in (2.1) and (2.4):

$$\sum_{n=0}^{\infty} n^N (n+a)^{-s} = \delta_{N0} a^{-s} + \sum_{k=0}^{\infty} \binom{-s}{k} a^k \zeta(s-N+k) \tag{2.9}$$

which leads to the special values

$$\begin{aligned}\sum_{n=0}^{\infty} n^N (n+a)^L \\ &= \delta_{N0} a^L + \sum_{k=0}^{L} \binom{L}{k} a^k \zeta(k-N-L) \\ &\quad + \left\{(-1)^{N+1} \frac{N!L!}{(N+L+1)!} a^{N+L+1}\right\}.\end{aligned} \tag{2.10}$$

The extra term in (2.10) is the $k = N+L+1$ term of (2.9). Generally, one can insert powers of the summation index under the sum with relative ease, as this example illustrates.

The second variation is the insertion of alternating signs under the sum:

$$\sum_{n=0}^{\infty} (-1)^{n+1} n^N (n+a)^{-s} = -\delta_{N0} a^{-s} + \sum_{k=0}^{\infty} \binom{-s}{k} a^k \eta(s-N+k) \tag{2.11}$$

where

$$\eta(s) \equiv \sum_{m=1}^{\infty} (-1)^{m+1} m^{-s} = (1-2^{1-s})\zeta(s) \tag{2.12}$$

is the alternating-sign Riemann ζ function. Because $\eta(s)$ has no pole for finite s, there are no extra terms generated by series commutation in the alternating-sign case [13-15]. Thus

$$\sum_{n=0}^{\infty} (-1)^{n+1} n^N (n+a)^L = -\delta_{N0} a^L + \sum_{k=0}^{L} \binom{L}{k} a^k \eta(k-N-L). \tag{2.13}$$

For $N=0$, the right-hand side of (2.13) equals $-E_L(a)/2$ [10] where $E_L(a)$ is the Lth Euler polynomial [7]. This is the alternating-sign equivalent of (2.2), defining the Bernoulli polynomials.

Finally, let us consider the Lerch function [7], obtained by inserting a factor z^n under the sum in (2.1);

$$\begin{aligned}\Phi(s, z, a) \equiv \sum_{n=0}^{\infty} z^n (n+a)^{-s} &= z^{-a} \sum_{k=0}^{\infty} \frac{1}{k!} (\ln z)^k \zeta(s-k, a) \\ &\quad + \{z^{-a} \Gamma(1-s)(-\ln z)^{s-1}\} \qquad |\ln z| < 2\pi.\end{aligned} \tag{2.14}$$

This function has poles at all positive integers $s=1, 2, 3, \ldots$, and is not a ζ function, but rather a Dirichlet series. The second equality is obtained by expanding the exponent in $z^n = z^{-a} \exp[(n+a) \ln z)]$ and commuting sums, with an extra term being generated as shown.

If we did not know (2.14), we could still evaluate the numerical sum at the point $s=-L$, $L=1, 2, 3, \ldots$:

$$\sum_{n=0}^{\infty} z^n (n+a)^L = z^{-a} \sum_{n=0}^{\infty} (n+a)^L \exp[(n+a) \ln z]$$

$$= z^{-a} \sum_{n=0}^{\infty} (n+a)^L \sum_{k=0}^{\infty} \frac{1}{k!} (\ln z)^k (n+a)^k$$

$$= z^{-a} \sum_{k=0}^{\infty} \frac{1}{k!} (\ln z)^k \zeta(-L-k, a) + \{(-1)^{L+1} z^{-a} \Gamma(L+1)(\ln z)^{-L-1}\}$$

$$L=1, 2, 3, \ldots; |a|<1; |\ln z|<2\pi. \tag{2.15}$$

Here the extra term is the $k=-(L+1)$ term and the result agrees with (2.14). So our device for finding extra terms again works; but unlike the previous examples, the sum Σ_k which gets extended here is an *infinite* sum over $k=0, 1, 2, \ldots$ (the extension being into the negative integers). This example also shows that the insertion of a power factor z^n under the sum of the Hurwitz ζ function changes the Bernoulli polynomial result (2.2) into an infinite series with Bernoulli polynomial coefficients. This type of modification in the value of a ζ function at $s=-np$, from polynomial to transcendental function, when extra functions are inserted under the sum, is typical of Dirichlet series. The factor z^n is representative of functions $f(n)$ of n, having infinite radius of convergence, which can be inserted under the sum, to convert ζ functions into Dirichlet series.

The other variations considered before, namely (i) $n+a \to n^\alpha + a$, (ii) insertion of n^N under the sum and (iii) insertion of alternating sign under the sum, can all be done with (2.14) and (2.15). New (or at any rate untabulated) results are readily obtained. However, Lerch's function is not our main focus here, so we proceed to more general matters.

2.2. General single-sum ζ functions

The ζ function

$$\sum_{m=1}^{\infty} (a_1 m^{\alpha_1} + \ldots + a_N m^{\alpha_N})^{-s} \qquad \alpha_1 > \alpha_2 > \ldots > \alpha_N > 0 \tag{2.16}$$

could be associated with an operator of order α_1, in which the lower-order terms represent 'perturbations' of the leading term. Generally it is possible to use the binomial expansion repeatedly—or the multinomial series once—to find an exact series for the ζ function (2.16) in powers of the parameters a_i. (One has to exercise care not to use the binomial series outside its radius of convergence. If one does, the resulting power series will be asymptotic. In other respects the procedure is straightforward.) One can then set $s=-L$ and obtain the polynomial value of the ζ function at these points.

(Note that, at these points, the asymptotic series resulting from misuse of the binomial expansion will yield the same polynomial result.)

A more direct way to evaluate (2.16) at $s=-L$ is

$$\sum_{m=1}^{\infty}(a_1 m^{\alpha_1}+\ldots+a_N m^{\alpha_N})^L$$

$$=\sum_{m=1}^{\infty}\sum_{k_1+\ldots+k_N=L}(L|k_1\ldots k_N)a_1^{k_1}\ldots a_N^{k_N}m^{\alpha_1 k_1+\ldots+\alpha_N k_N}$$

$$=\sum_{k_1+\ldots+k_N=L}(L|k_1\ldots k_N)a_1^{k_1}\ldots a_N^{k_N}\zeta(-\alpha_1 k_1-\ldots-\alpha_N k_N)+\{\ \} \qquad (2.17)$$

where the multinomial series is used and $(L|k_1\ldots k_N)$ are the multinomial coefficients.

In those cases where $\{\}=0$, (2.17) is the final result. This extremely simple derivation works as well as it does because one is computing a polynomial, and not an infinite series. When the multinomial series is finite, there are no convergence problems arising from it—the relative size of terms in $(a_1 m^{\alpha_1}+\ldots)^L$ is irrelevant. Away from the points $s=-L$, the multinomial series becomes infinite and its convergence becomes a major consideration.

The conditions under which there will be an extra term in (2.17) have been spelled out previously. If it is possible to extend the sum over $k_1, k_2, \ldots, k_N$ to other integer values, such that $\zeta(1)$ can be made to appear in (2.17), then such terms will contribute to $\{\}$. If this is not possible, then $\{\}=0$ in (2.17).

Variations of the problem (2.17) along the lines considered in § 2.1 are easy to deal with.

(i) Insertion of a power m^{β} under the sum in (2.17) merely shifts the argument of the Riemann ζ function in the answer by $-\beta$, much as in (2.10).

(ii) Insertion of the alternating-sign factor $(-1)^{m+1}$ under the sum in (2.17) causes $\zeta(-\alpha_1 k_1-\ldots-\alpha_N k_N)$ to be replaced by $\eta(-\alpha_1 k_1-\ldots-\alpha_N k_N)$. As $\eta(s)$ has no pole for finite s, there will be no extra term.

(iii) Much as in (2.15)

$$\sum_{m=1}^{\infty}z^m(a_1 m^{\alpha_1}+\ldots+a_N m^{\alpha_N})^L$$

$$=\sum_{k=0}^{\infty}\frac{1}{k!}(\ln z)^k\sum_{m=1}^{\infty}m^k(a_1 m^{\alpha_1}+\ldots+a_N m^{\alpha_N})^L$$

where (2.17) and comment (i) evaluate Σ_m on the right.

3. Multidimensional ζ functions

The binomial or multinomial series makes it relatively easy to compute single-sum ζ functions in terms of the Riemann ζ function, for arbitrary s. This approach does not work for multidimensional ζ functions, because the relative size of different summation indices is completely arbitrary. Hence there is no systematic way to use the binomial series within its radius of convergence, which parallels, for example, (2.1). However, at those points (1.7) along the real axis which are singled out in the present paper, one is dealing with *finite*, not infinite, binomial series, and the relative size of different summation indices is irrelevant. This makes it easy to extend the considerations in § 2 to the multidimensional case. Indeed, our method is much more powerful (relative to what one can achieve for arbitrary s) in multidimensional problems than it is in the single-sum case.

In this section, we consider three categories of multiple-sum ζ functions.

(a) Linear ζ functions, for which many exact results are known [11, 16], and which therefore provide a test of our procedure.

(b) Epstein or quadratic ζ functions [9]. These are rather trivial to evaluate at the non-positive integers by our method.

(c) Zeta functions involving arbitrary powers of the various summation indices. Such ζ functions have not been studied in any context to the author's knowledge. We shall obtain quite explicit results for them.

3.1. *Linear ζ functions*

Consider the simplest linear ζ function [11]

$$\sum_{m,n=1}^{\infty} (m+n)^{-s} = \zeta(s-1) - \zeta(s). \tag{3.1}$$

Evaluating the left-hand side of (3.1) at $s=-(2N+1)$ yields

$$\begin{aligned}\sum_{m,n=1}^{\infty} (m+n)^{2N+1} &= \sum_{k=0}^{2N+1} \binom{2N+1}{k} \zeta(-k)\zeta(k-2N-1) \\ &= 2\zeta(0)\zeta(-2N-1) = -\zeta(-2N-1) \end{aligned}\tag{3.2}$$

in agreement with the right-hand side of (3.1). Note that there is no extra term: extending Σ_k to $k=2N+2$, to pick up the $\zeta(1)$ term, yields zero. This is because $\zeta(1)$ is multiplied by two zeros: a vanishing binomial coefficient and $\zeta(-2N-2)$.

Evaluating the sum (3.1) at $s=-2N$ gives

$$\begin{aligned}\sum_{m,n=1}^{\infty} (m+n)^{2N} &= \zeta(-2N-1) \\ &= \sum_{k=1}^{2N-1} \binom{2N}{k} \zeta(-k)\zeta(k-2N) + \left\{-\frac{2}{2N+1}\zeta(-2N-1)\right\}. \end{aligned}\tag{3.3}$$

Here all the terms with odd k contribute. Also, there is an extra term coming from $k=-1$ and $k=2N+1$, as shown. The right-hand side of (3.3) must sum to $\zeta(-2N-1)$. One verifies that it does.

Consider another known result [16]:

$$\sum_{m,n=1}^{\infty} (m+2n)^{-s} = \tfrac{1}{2}\zeta(s-1) - \tfrac{1}{2}\zeta(s)(1+2^{-s}). \tag{3.4}$$

For $s=-(2N+1)$, one readily verifies that the left-hand side agrees with the right, and that there is no extra term. For $s=-2N$:

$$\begin{aligned}\sum_{m,n=1}^{\infty} (m+2n)^{2N} &= \tfrac{1}{2}\zeta(-2N-1) \\ &= \sum_{k=1}^{2N-1} \binom{2N}{k} 2^{2N-k} \zeta(-k)\zeta(k-2N) \\ &\quad + \left\{-\frac{1}{2N+1}(2^{2N+1}+\tfrac{1}{2})\zeta(-2N-1)\right\}. \end{aligned}\tag{3.5}$$

Again the extra term comes from $k=-1$ and $k=2N+1$. It is not difficult to check that (3.5) is indeed an identity.

Proceeding now to the general case, we have

$$\sum_{m_i=1}^{\infty} (a_1 m_1 + \ldots + a_N m_N)^L$$
$$= \sum_{k_1+\ldots+k_N=L} (L|k_1 \ldots k_N) a_1^{k_1} \ldots a_N^{k_N} \zeta(-k_1) \ldots \zeta(-k_N) + \{\ \} \tag{3.6}$$

where the procedure to follow in calculating the extra term is clear. Exact results on linear ζ functions (for arbitrary s) much more general than (3.1) and (3.4) are available [16], and elaborate examples could be provided. We state without proof that the usual variations, of inserting powers of m_i and/or alternating signs under the sum in (3.6), are readily accomplished.

3.2. *Epstein ζ functions*

The simplest Epstein ζ function [9] is [17]

$$\sum_{m,n=-\infty}^{\infty}{}' (m^2+n^2)^{-s} = 4\zeta(s)\beta(s) \tag{3.7}$$

$$\beta(s) \equiv \sum_{n=0}^{\infty} (-1)^n (2n+1)^{-s} \tag{3.8}$$

where the prime means $m=n=0$ is excluded from the sum. In very special cases, such as this one, Epstein ζ functions can be simply expressed in terms of $\zeta(s)$ and other one-dimensional (1D) series like (3.8). A compendium of such results, with earlier references, can be found in [18]. Aside from these special cases, one has no idea at present how to express general Epstein ζ functions in terms of 1D sums for arbitrary s. However, at the negative integers, this is easy to do, as was first pointed out in [19].

The main points are well illustrated by the ζ function (3.7). Evaluating the left-hand series at $s=-L$ gives

$$\begin{aligned} \sum_{m,n=-\infty}^{\infty}{}' (m^2+n^2)^L &= \sum_{m,n=-\infty}^{\infty}{}' \sum_{k=0}^{L} \binom{L}{k} m^{2k} n^{2L-2k} \\ &= 4\sum_{k=0}^{L} \binom{L}{k} \zeta(-2k)\zeta(2k-2L) + 4\zeta(-2L) \\ &= 0 \qquad L=1,2,3,\ldots. \end{aligned} \tag{3.9}$$

Every term in the final result vanishes. There is no extra term, because there is no way to pick up a $\zeta(1)$ contribution—all Riemann ζ functions have arguments which are even integers. Equation (3.9) agrees with (3.7), because $\zeta(-L)=0$ for even L while $\beta(-L)=0$ for odd L. Some more information on $\beta(s)$ is

$$\begin{aligned} \beta(-L) &= \tfrac{1}{2}E_L \qquad L=0,1,2,\ldots \\ &= 0 \qquad L=1,3,5,\ldots \end{aligned} \tag{3.10}$$

where the E_L are the Euler numbers [8]. The point $s=0$ is also easy to deal with:

$$\begin{aligned} \sum_{m,n=-\infty}^{\infty}{}' (m^2+n^2)^0 &= 4\sum_{m,n=1}^{\infty} (m^2+n^2)^0 + 4\sum_{m=1}^{\infty} (m^2)^0 \\ &= 4\zeta(0)\zeta(0) + 4\zeta(0) = -1 \end{aligned} \tag{3.11}$$

since $\zeta(0) = -\frac{1}{2}$, in agreement with (1.11c) and (3.7).

As a further illustration of (1.11) consider the following double series, with a, b not both integers and $L=0, 1, 2, \ldots$:

$$\begin{aligned}
&\sum_{m,n=-\infty}^{\infty} [(m+a)^2+(n+b)^2]^L \exp[\mathrm{i}(mc+nd)] \\
&\qquad = \sum_{m,n=-\infty}^{\infty} \exp[\mathrm{i}(mc+nd)] \sum_{k=0}^{L} \binom{L}{k} (m+a)^{2k}(n+b)^{2L-2k} \\
&\qquad = \sum_{k=0}^{L} \binom{L}{k} \sum_{p=0}^{2k} \binom{2k}{p} a^{2k-p} \sum_{q=0}^{2L-2k} \binom{2L-2k}{q} \\
&\qquad\quad \times b^{2L-2k-q} \sum_{m,n=-\infty}^{\infty} m^p n^q \exp[\mathrm{i}(mc+nd)] \\
&\qquad = 0 \qquad a, b \text{ not both integers; } L=0, 1, 2, \ldots.
\end{aligned} \tag{3.12}$$

This vanishing result is assured by

$$\sum_{m,n=-\infty}^{\infty} m^p n^q \exp[\mathrm{i}(mc+nd)] = 0 \tag{3.13}$$

for all integers $p, q \geqslant 0$:

$$\begin{aligned}
&\sum_{m,n=-\infty}^{\infty} \exp[\mathrm{i}(mc+nd)] \begin{Bmatrix} m^{2M}n^{2N} \\ m^{2M}n^{2N-1} \\ m^{2M-1}n^{2N-1} \\ m^{2M} \\ m^{2M-1} \\ 1 \end{Bmatrix} \\
&\qquad = \sum_{m,n=1}^{\infty} \begin{Bmatrix} m^{2M}n^{2N} 4\cos(mc)\cos(nd) \\ m^{2M}n^{2N-1} 4\mathrm{i}\cos(mc)\sin(nd) \\ -m^{2M-1}n^{2N-1} 4\sin(mc)\sin(nd) \\ m^{2M} 2\cos(mc)[2\cos(nd)+\delta_{n1}] \\ m^{2M-1} 2\mathrm{i}\sin(mc)[2\cos(nd)+\delta_{n1}] \\ [2\cos(mc)+\delta_{m1}][2\cos(nd)+\delta_{n1}] \end{Bmatrix} \\
&\qquad = 0.
\end{aligned} \tag{3.14}$$

Here M, $N \geqslant 1$ are integers and we use

$$\sum_{m=1}^{\infty} \begin{Bmatrix} m^{2M}\cos(mx) \\ m^{2M-1}\sin(mx) \\ \cos(mx) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -\frac{1}{2} \end{Bmatrix}. \tag{3.15}$$

The latter formulae are derived in detail in [13]. Both (3.14) and (3.15) are special values of Dirichlet series defined throughout the s plane. ([13] is a compendium of a large number of results on Dirichlet series.)

When a, b are both integers, the term $m=-a$, $n=-b$ on the left-hand side vanishes for $L>0$, while for $L=0$ it is non-vanishing and equal to $\exp[\mathrm{i}(-ac-bd])$. This term is simply moved to the right-hand side to obtain (1.11c).

The calculation summarised by (3.12) can obviously be generalised to N dimensions. Equation (3.13) is readily extended to N dimensions:

$$\sum_{m_i=-\infty}^{\infty} m_1^{M_1}\dots m_N^{M_N}\exp[\mathrm{i}(m_1h_1+\dots+m_Nh_N)]=0 \tag{3.16}$$

where the M_i are non-negative integers. We feel that further discussion of this point is unnecessary, and that we have given an essentially complete derivation of (1.11) which is completely different from Epstein's. An advantage of the new derivation is its flexibility. It can be used to extend (1.11) in directions where Epstein's ζ function arguments cannot readily follow.

Consider the insertion of any positive integral powers of m and n under the sum in (3.12). This merely shifts m^p and n^q under the sum Σ_{mn} in the second equality of (3.12) to higher powers, and the result remains zero because of (3.13). The same thing is true for any dimension N because of (3.16). (Note that alternating sign has already been incorporated into (1.11) in the phase factor and need not be considered separately.)

A slightly different calculation is

$$\begin{aligned}\sum_{m,n=1}^{\infty} m^M n^N (am^2+bn^2)^L &= \sum_{k=0}^{L}\binom{L}{k}a^k b^{L-k}\zeta(-2k-M)\zeta(2k-2L-N)+\{\ \}\\ &=0 \qquad \text{unless both } M, N \text{ are odd.}\end{aligned} \tag{3.17}$$

Here each term vanishes separately, including $\{\}$, unless both M and N are odd integers. When M, N are odd, the terms in Σ_k are non-zero, as is the extra term coming from $k=L+(N+1)/2$ and $k=-(M+1)/2$:

$$\left\{(-1)^{(M+1)/2}\frac{L![(M-1)/2]!}{2[L+(M+1)/2]!}a^{-(M+1)/2}b^{L+(M+1)/2}\right.$$
$$\left.\times\zeta(-M-N-1-2L)+M\leftrightarrow N,\, a\leftrightarrow b\right\}. \tag{3.18}$$

When the sum in (3.17) is extended to $-\infty\leqslant m, n\leqslant\infty$, the result trivially vanishes, as it should.

3.3. More general ζ functions

The preceding study of linear and Epstein ζ functions reveals that far more general multidimensional ζ functions yield to the same method. For example, as a generalisation of (3.6) we have

$$\begin{aligned}\sum_{m_i=1}^{\infty}(a_1m_1^{\alpha_1}+\dots+a_Nm_N^{\alpha_N})^L &= \sum_{k_1+\dots+k_N=L}(L|k_1\dots k_N)a_1^{k_1}\dots a_N^{k_N}\\ &\quad\times\zeta(-\alpha_1k_1)\dots\zeta(-\alpha_Nk_N)+\{\} \qquad \alpha_i>0.\end{aligned} \tag{3.19}$$

Suppose that the α_i are even integers and $L>0$. Then every term in the multinomial series (3.19) contains one or more factors of $\zeta(-2n)=0$, and consequently vanishes. Moreover, there is no extra term. Thus the special value (3.19) is zero for $L=1,2,3,\dots$, and $\alpha_i=2M_i$, much as for an Epstein ζ function.

When one or more of the $\alpha_i = 1$, it becomes possible to pick up $\zeta(1)$ contributions to the extra term, as we have seen in the case of linear ζ functions. For brevity, we omit a discussion of examples of this type, as things are much as in the linear case.

Suppose that $\alpha_i > 1$. Then in (3.19) it is impossible to extend the multinomial sum over the integers and find $\zeta(1)$ terms, so $\{\} = 0$. Of course, the individual terms of the multinomial series do not vanish, and in general the special value (3.19) is non-zero.

4. Derivatives of ζ functions

Derivatives of ζ functions are no longer ζ functions, but rather Dirichlet series with logarithmic factors under the sum:

$$\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^n Z(s) = (-1)^n \sum_m (\ln \lambda_m)^n \lambda_m^{-s}. \tag{4.1}$$

The method of this paper can be used to evaluate such series at the same points (1.7) at which $Z(s)$ is easily evaluated. This may not seem surprising, but neither is it entirely obvious. To fully describe the details would require another paper. Here we merely sketch the procedure, using single-sum ζ functions as an example. In general, the derivative (4.1) evaluated at $s = -np$ is not a polynomial in the parameters of λ_m, as is $Z(-np)$, but rather a transcendental function in these parameters.

Because the series (4.1) is more difficult to evaluate than the undifferentiated series $Z(s)$, it becomes all the more important to study known examples. For this reason we return to the Hurwitz ζ function. From (2.1) one finds, after some work [10],

$$\begin{aligned}\zeta'(-L, a) - a^L \ln a &= \sum_{k=0}^{L} \binom{L}{k} a^k \{[\psi(k-L) - \psi(-L)]\zeta(k-L) + \zeta'(k-L)\} \\ &\quad + \left\{\frac{a^{L+1}}{L+1}(C_L - \gamma)\right\}_1 + \left\{\sum_{k=L+2}^{\infty} (-1)^{L+k} \frac{L!(k-L-1)!}{k!} a^k \zeta(k-L)\right\}_2 \\ &\quad |a| < 1\end{aligned} \tag{4.2}$$

where $C_L = 1 + \frac{1}{2} + \ldots + 1/L$ and γ is the Euler constant. Equation (4.2) is transcendental, as promised. The extra term has been split into two parts: $\{\}_1$ originates from the pole in $\zeta(s)$ at $s = 1$, as did all the extra terms in previous sections. $\{\}_2$ arises from the pole in $\psi(k-L) - \psi(-L)$ for $k \geq L+2$, and *not* from the pole in $\zeta(s)$. Our task now is to obtain (4.2) without the help of (2.1).

From the definition of $\zeta(s, a)$ we have

$$\begin{aligned}\zeta'(-L, a) - a^L \ln a &= -\sum_{m=1}^{\infty} \ln(m+a)(m+a)^L \\ &= -\sum_{k=0}^{L} \binom{L}{k} a^k \sum_{m=1}^{\infty} m^{L-k}[\ln m + \ln(1 + a/m)] + \{\}_1 + \{\}_2 \\ &= \sum_{k=0}^{L} \binom{L}{k} a^k \left\{\zeta'(k-L) + \sum_{r=1}^{\infty} (-1)^r \frac{a^r}{r} \zeta(k+r-L)\right\} \\ &\quad + \{\ \}_1 + \{\ \}_2 \qquad |a| < 1\end{aligned} \tag{4.3}$$

where the extra terms $\{\,\}_{1,2}$ will be calculated shortly. Note that the expansion $\ln(1+a/m)=a/m-(a/m)^2/2+\ldots$ has been used and

$$\zeta'(s)=-\sum_{m=1}^{\infty}\ln m/m^s$$

is the derivative of $\zeta(s)$. The finite sums $\Sigma_{k=0}^{L}$ in (4.2) and (4.3) are seen to be identical, because of the identity

$$\sum_{\substack{k\geq 0\\ r\geq 1\\ k+r=N}}\binom{-s}{k}(-1)^r\frac{1}{r}=\binom{-s}{N}\sum_{p=0}^{N-1}\frac{1}{s+p}$$

$$=\binom{-s}{N}[\psi(s+N)-\psi(s)]. \tag{4.4}$$

What remains is to show that the extra terms $\{\,\}_{1,2}$ can be routinely evaluated. To find $\{\,\}_1$ (which is associated with the pole in $\zeta(s)$ at $s=1$) we set $s=-L+\varepsilon$ rather than $s=-L$ in (4.3). Then, to pick up the $\zeta(1)$ contribution, we set $k=L+1$ in the term containing $\zeta'(k-L+\varepsilon)$, and $k+r=L+1$ in the term containing $\zeta(k+r-L+\varepsilon)$. This gives

$$\begin{aligned}\{\,\}_1&=\binom{L-\varepsilon}{L+1}a^{L+1}\zeta'(1+\varepsilon)+\sum_{k+r=L+1}\binom{L-\varepsilon}{k}a^{L+1}(-1)^r\frac{1}{r}\zeta(1+\varepsilon)\\
&=\binom{L-\varepsilon}{L+1}a^{L+1}\left\{\zeta'(1+\varepsilon)+\zeta(1+\varepsilon)\sum_{p=0}^{L-1}\frac{1}{p-L+\varepsilon}\right\}\\
&=\binom{L-\varepsilon}{L+1}a^{L+1}\left\{\left(-\frac{1}{\varepsilon^2}+\text{constant}+\mathrm{O}(\varepsilon)\right)\right.\\
&\quad\left.+\left(\frac{1}{\varepsilon}+\gamma+\mathrm{O}(\varepsilon)\right)\left(\frac{1}{\varepsilon}-C_L+\mathrm{O}(\varepsilon)\right)\right\}\\
&=\frac{a^{L+1}}{(L+1)\Gamma(-\varepsilon)}\left\{\frac{1}{\varepsilon}(\gamma-C_L)+\text{constant}+\mathrm{O}(\varepsilon)\right\}\\
&=\frac{a^{L+1}}{L+1}(C_L-\gamma)\end{aligned}\tag{4.5}$$

which agrees with (4.2). Note that (4.4) was used to obtain the first equality. As always, the limit $\varepsilon\to 0$ is understood.

The other extra term $\{\,\}_2$ is not associated with the pole in $\zeta(s)$ at $s=1$. Let us replace $s=-L$ by $s=-L+\varepsilon$ and write the double sum $\Sigma_{k,r}$ in (4.3) as

$$\sum_{k,r}\binom{L-\varepsilon}{k}a^{k+r}(-1)^r\frac{1}{r}\zeta(k+r-L+\varepsilon)$$

$$=\sum_{n}a^n\zeta(n-L+\varepsilon)\binom{L-\varepsilon}{n}\sum_{p=0}^{n-1}\frac{1}{p-L+\varepsilon}. \tag{4.6}$$

For $n-1\geq L$, the sum over p is singular:

$$\sum_{p=0}^{n-1}\frac{1}{p-L+\varepsilon}=\frac{1}{\varepsilon}+\text{finite}\qquad n-1\geq L.$$

The $1/\varepsilon$ term cancels the vanishing binomial coefficient and yields a contribution, which is $\{\}_2$. Clearly $n-1=L$ just gives us the ζ-function pole term $\zeta(1+\varepsilon)$, which already went into the extra term (4.5). So we begin the sum over n at $n=L+2$ and the right-hand side of (4.6) becomes

$$\{\}_2=\sum_{n=L+2}^{\infty} a^n\zeta(n-L)\binom{L-\varepsilon}{n}\left(\frac{1}{\varepsilon}+\text{finite}\right) \tag{4.7}$$

which is easily seen to agree with (4.2).

To extend these considerations to ζ functions constructed from eigenvalues λ_m which are polynomials of order N in the summation index m, one has the fundamental theorem of algebra:

$$\lambda_m=A(m-b_1)\ldots(m-b_N).$$

Then

$$\sum_m \lambda_m^{-s}\ln\lambda_m=\sum_m \lambda_m^{-s}\left\{\ln A+N\ln m-\sum_{r=1}^{\infty}\frac{1}{rm^r}(b_1^r+\ldots+b_N^r)\right\}.$$

Setting $s=-L$ there remains the evaluation of the sums

$$\sum_m\begin{Bmatrix}\ln m\\ m^{-r}\end{Bmatrix}(m-b_1)^L\ldots(m-b_N)^L$$

both of which can be handled by techniques developed earlier in this paper.

$\zeta'(s,a)$ and the Lerch function $\Phi(s,a,z)$ in (2.14) are two of the Dirichlet series associated with the Hurwitz ζ function $\zeta(s,a)$. We have shown that both Dirichlet series can be evaluated by our method at the points $s=-L$ where $\zeta(s,a)$ becomes the Bernouilli polynomial (2.2). While not a proof, this strongly indicates that many, if not all, Dirichlet series associated with $\zeta(s,a)$ can be routinely evaluated by our method at $s=-L$, and that the same is true for more general ζ functions $Z(s)$ and their associated Dirichlet series $D(s)$ at the points (1.7).

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